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VECTOR & SCALAR

FOR MALAYSIAN POLYTECHNIC STUDENTS

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VECTOR AND SCALAR FOR MALAYSIAN POLYTECHNIS STUDENTS eBook.

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Thank you.

ABSTRACT

VECTOR AND SCALAR
FOR MALAYSIAN POLYTECHNIS STUDENTS

This eBook takes the reader on a journey to explore fundamental concepts in mathematics that are relevant in the fields of science and technology: vectors and scalars.

This eBook contains FOUR (4) topics, through clear explanations and an easy-to-understand approach. The reader will be taken from basic understanding to more complex applications of these two concepts. With rich illustrations and related exercises, readers will be invited to explore various applications of vectors and scalars, deepen their understanding, and expand their mathematical thinking.

Overall, this book aims to provide an informative guide for those who wish to improve their understanding of vectors and scalars.

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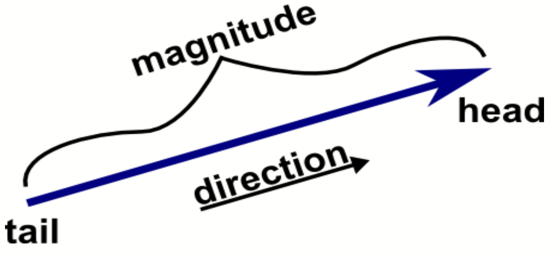
2. CALCULATION OF MAGNITUDE VECTOR AND UNIT VECTOR

BASIC VECTOR DEFINITION

VECTOR NOTATION

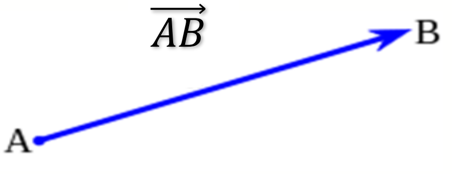
VECTOR is a quantity that has both **magnitude** and **direction**.

- Example: displacement, velocity, acceleration, and force.



The length of the line shows its **magnitude** and the arrowhead points in the **direction**.

A **vector** from A to B can be denoted as $\overrightarrow{AB}$, $\underline{a}$ (underline vectors) or $\boldsymbol{a}$ (bold typeface).



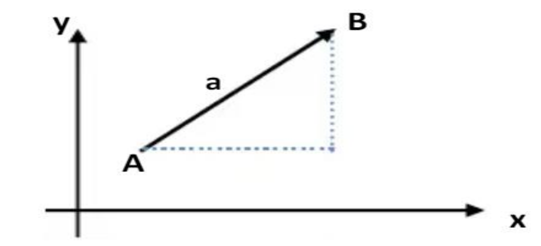
Vector Arrow carry a point A to point B.

SCALAR is a quantity that has **magnitude** but no direction.

- Example: distance, height and temperature.

VECTOR REPRESENTATION

A vector can be represented by a line segment with an arrow, known as a **directed line segment**.

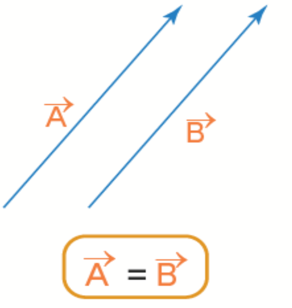


$\left|\overrightarrow{AB}\right| = |\boldsymbol{a}| =$ magnitude AB

- The **<u>arrowhead</u>** represents the **direction** of the vectors.
- The **<u>length</u>** of the line represents the **magnitude** of the vectors.

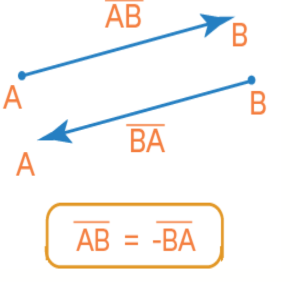
EQUAL VECTOR

- Two vectors are said to be equal if they have the **same magnitude and direction.**
- Vector **A** is said to be an equal vector to vector **B** if they have the same length and are pointing in the same direction.



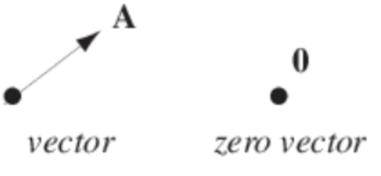
NEGATIVE VECTOR

- The negative vector of $\overrightarrow{\mathbf{AB}}$ is $\overrightarrow{\mathbf{BA}}$.
- $\overrightarrow{\mathbf{BA}}$ has the same magnitude as $\overrightarrow{\mathbf{AB}}$ but opposite direction.



ZERO VECTOR

- Also known as null vector, denoted 0, has a magnitude equal to 0, and thus has all components equal to 0.



CALCULATION OF MAGNITUDE VECTOR AND UNIT VECTOR

MAGNITUDE VECTOR

- The **magnitude** of a vector **AB** is the length of the vector **AB** and is denoted by **AB**.
- Formulas for the magnitude of vectors in 2D and 3D dimensions in terms of their coordinates are derived by:

Vector $\vec{v} = (a, b)$

$$|\vec{v}| = \sqrt{a^2 + b^2}$$

Vector $\vec{v} = (a, b, c)$

$$|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$$

UNIT VECTOR

- A **unit vector** is a vector with a magnitude of 1. Unit vectors are marked with a cap symbol, which looks like a little arrow pointing upward: ^.
- To calculate a unit vector ($\hat{v}$), divide the vector $\vec{v}$ by its magnitude $|\vec{v}|$. In other words, follow this formula:

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

- $\hat{v}$ is the unit vector
- $\vec{v}$ is the vector
- $|\vec{v}|$ is the magnitude

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EXAMPLE 1

Find the magnitude and unit vector of vectors as below.

i. Vector $\vec{\boldsymbol{v}} = 3\boldsymbol{i} + 4\boldsymbol{j}$

ii. Vector $\vec{\boldsymbol{v}} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{2k}$

Solutions:

i. Vector $\vec{\boldsymbol{v}} = 3\boldsymbol{i} + 4\boldsymbol{j}$

Using magnitude formula,

$$|\vec{\boldsymbol{v}}| = \sqrt{\boldsymbol{a}^2 + \boldsymbol{b}^2}$$
$$= \sqrt{3^2 + 4^2}$$
$$= \sqrt{9 + 16}$$
$$= \sqrt{25} = 5$$

Using unit vector formula,

$$\hat{\boldsymbol{v}} = \frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}$$
$$= \frac{3\boldsymbol{i} + 4\boldsymbol{j}}{5}$$
$$= \frac{3\boldsymbol{i}}{5} + \frac{4\boldsymbol{j}}{5}$$

ii. Vector $\vec{\boldsymbol{v}} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{2k}$

Using magnitude formula,

$$|\vec{\boldsymbol{v}}| = \sqrt{\boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2}$$
$$= \sqrt{2^2 + (-1)^2 + (-2)^2}$$
$$= \sqrt{4 + 1 + 4}$$
$$= \sqrt{9} = 3$$

Using unit vector formula,

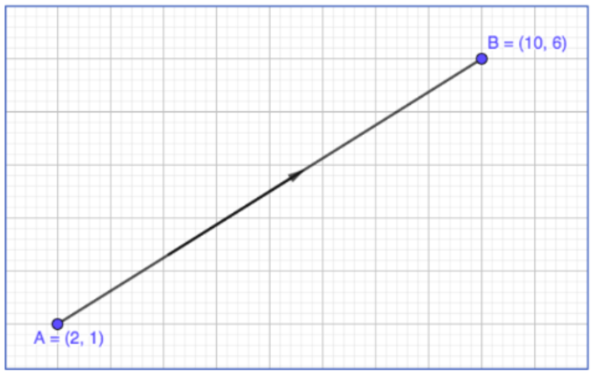
$$\hat{\boldsymbol{v}} = \frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}$$
$$= \frac{2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}}{\boldsymbol{3}}$$
$$= \frac{2\boldsymbol{i}}{3} - \frac{\boldsymbol{j}}{3} - \frac{2\boldsymbol{k}}{3}$$

EXAMPLE 2

Based on vector in Cartesian plane as shown below.

a. Write a vector $\overrightarrow{\mathbf{AB}}$ in terms of $\boldsymbol{ai} + \boldsymbol{bj}$ and $\binom{a}{b}$

b. Find the magnitude and unit vector of $\overrightarrow{\mathbf{AB}}$



Solutions:

Write vector $\overrightarrow{\mathbf{AB}}$ in terms of $\boldsymbol{ai} + \boldsymbol{bj}$ and $\binom{a}{b}$.

$$\overrightarrow{\mathbf{AB}} = \mathbf{B} - \mathbf{A}$$

$$\boldsymbol{ai} + \boldsymbol{bj} = (10 - 2)\boldsymbol{i} + (6 - 1)\boldsymbol{j}$$
$$= 8\boldsymbol{i} + 5\boldsymbol{j}$$

$$\binom{\boldsymbol{a}}{\boldsymbol{b}} = \binom{10}{6} - \binom{2}{1}$$
$$= \binom{\mathbf{8}}{\mathbf{5}}$$

Find the magnitude and unit vector of $\overrightarrow{\mathrm{AB}}$.

$$|\overrightarrow{\mathbf{AB}}| = \sqrt{\boldsymbol{a}^2 + \boldsymbol{b}^2}$$
$$= \sqrt{8^2 + 5^2}$$
$$= \sqrt{64 + 25}$$
$$= \sqrt{89}$$

$$\widehat{\mathbf{AB}} = \frac{\overrightarrow{\mathbf{AB}}}{|\overrightarrow{\mathbf{AB}}|}$$
$$= \frac{8\boldsymbol{i} + 5\boldsymbol{j}}{\sqrt{89}}$$
$$= \frac{8\boldsymbol{i}}{\sqrt{89}} + \frac{5\boldsymbol{j}}{\sqrt{89}}$$

EXERCISE

a) Find the magnitude of the vector $\boldsymbol{a} = 5\boldsymbol{i} + 3\boldsymbol{j}$.

b) If $|\overrightarrow{\mathbf{AB}}| = 5 \text{ units}$, find the magnitude of the vectors $\mathbf{3}\overrightarrow{\mathbf{AB}}$ and $-\mathbf{5}\overrightarrow{\mathbf{AB}}$.

c) Find the magnitude of the vector $\overrightarrow{\mathbf{AB}}$ if $\overrightarrow{\mathbf{OA}} = 2\mathbf{i} + \mathbf{j}$ and $\overrightarrow{\mathbf{OB}} = -4\mathbf{i} + 7\mathbf{j}$.

d) Find $|\overrightarrow{\mathbf{PQ}}|$ if the position vector of $\boldsymbol{p} = \boldsymbol{i} - 5\boldsymbol{j}$ and position vector $\boldsymbol{q} = 7\boldsymbol{i} + 5\boldsymbol{j}$.

02 The Operations on Vectors

1 CALCULATION OF VECTOR ADDITION AND SUBTRACTION USING

A) PARALLELOGRAM LAW OF ADDITION

B) TRIANGLE LAW OF ADDITION

OPERATIONS ON VECTOR

ADDITION OF VECTORS

For any vector $\boldsymbol{v_1} = a_1\boldsymbol{i} + b_1\boldsymbol{j}$ and $\boldsymbol{v_2} = a_2\boldsymbol{i} + b_2\boldsymbol{j}$

- The addition of two parallel vectors, $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$, can be written as, $\boldsymbol{v_1} + \boldsymbol{v_2}$.
- The result of this addition is a vector which is called the **resultant vector**.
- A resultant vector is the combination of two or more single vectors.

When **two vectors** with the **same direction** is added up, the resultant vector has

- the same direction with both the vectors.
- a magnitude **equal to the sum** of the magnitudes of both the vectors.

ADDITION

$$\boldsymbol{v_1} + \boldsymbol{v_2} = (a_1 + a_2)\boldsymbol{i} + (b_1 + b_2)\boldsymbol{j}$$

SUBTRACTION OF VECTORS

For any vector $\boldsymbol{v_1} = a_1\boldsymbol{i} + b_1\boldsymbol{j}$ and $\boldsymbol{v_2} = a_2\boldsymbol{i} + b_2\boldsymbol{j}$

- The subtraction of two parallel vectors, $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$, is the sum of vector $\boldsymbol{v_1}$ and negative vector $\boldsymbol{v_2}$, that can be written as, $\boldsymbol{v_1} + (-\,\boldsymbol{v_2})$.

SUBTRACTION

$$\boldsymbol{v_1} - \boldsymbol{v_2} = (a_1 + a_2)\boldsymbol{i} + (-(b_1 + b_2)\boldsymbol{j})$$

EXAMPLE 3

If $\boldsymbol{u} = 4\boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{v} = 2\boldsymbol{i} + 6\boldsymbol{j}$. Find,

i. $u + v$

ii. $u - v$

iii. $3u + 2v$

Solutions:

$$
\begin{aligned}
u + v &= (4\boldsymbol{i} - 3\boldsymbol{j}) + (2\boldsymbol{i} + 6\boldsymbol{j}) \\
&= (4 + 2)\boldsymbol{i} + (-3 + 6)\boldsymbol{j} \\
&= 6\boldsymbol{i} + 3\boldsymbol{j}
\end{aligned}
$$

$$
\begin{aligned}
u - v &= (4\boldsymbol{i} - 3\boldsymbol{j}) + (-(2\boldsymbol{i} + 6\boldsymbol{j})) \\
&= (4 - 2)\boldsymbol{i} + (-3 - 6)\boldsymbol{j} \\
&= 2\boldsymbol{i} + (-9\boldsymbol{j}) \\
&= 2\boldsymbol{i} - \mathbf{9}\boldsymbol{j}
\end{aligned}
$$

$$
\begin{aligned}
3u + 2v &= 3(4\boldsymbol{i} - 3\boldsymbol{j}) + 2(2\boldsymbol{i} + 6\boldsymbol{j}) \\
&= (12\boldsymbol{i} - 9\boldsymbol{j}) + (4\boldsymbol{i} + 12\boldsymbol{j}) \\
&= (12 + 4)\boldsymbol{i} + (-9 + 12)\boldsymbol{j} \\
&= 16\boldsymbol{i} + 3\boldsymbol{j}
\end{aligned}
$$

EXERCISE

a) $3\boldsymbol{u} - 2\boldsymbol{v}$, where $\boldsymbol{u} = 4\boldsymbol{i} + 8\boldsymbol{j} + 12\boldsymbol{k}$ and $\boldsymbol{v} = 3\boldsymbol{i} + 6\boldsymbol{j} + 9\boldsymbol{k}$.

b) $\boldsymbol{u} + 4\boldsymbol{v}$, where $\boldsymbol{u} = 8\boldsymbol{i} - 15\boldsymbol{j}$ and $\boldsymbol{v} = -\boldsymbol{i} + 5\boldsymbol{j}$.

ADDITION OF NON-PARALLEL VECTORS

Addition of two non-parallel vectors, can be shown by using two laws.

1. Parallelogram Law of Addition
2. Triangle Law of Addition

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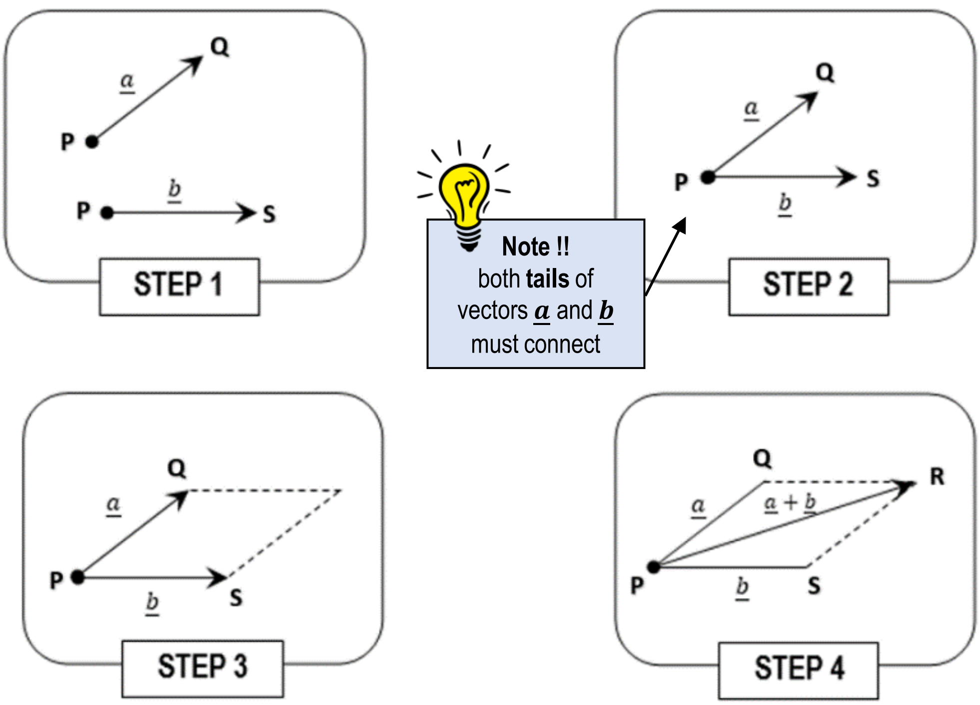
PARALLELOGRAM LAW OF ADDITION

The sum of the two vectors is given by the diagonal of the parallelogram passing through the tail of the two vectors.



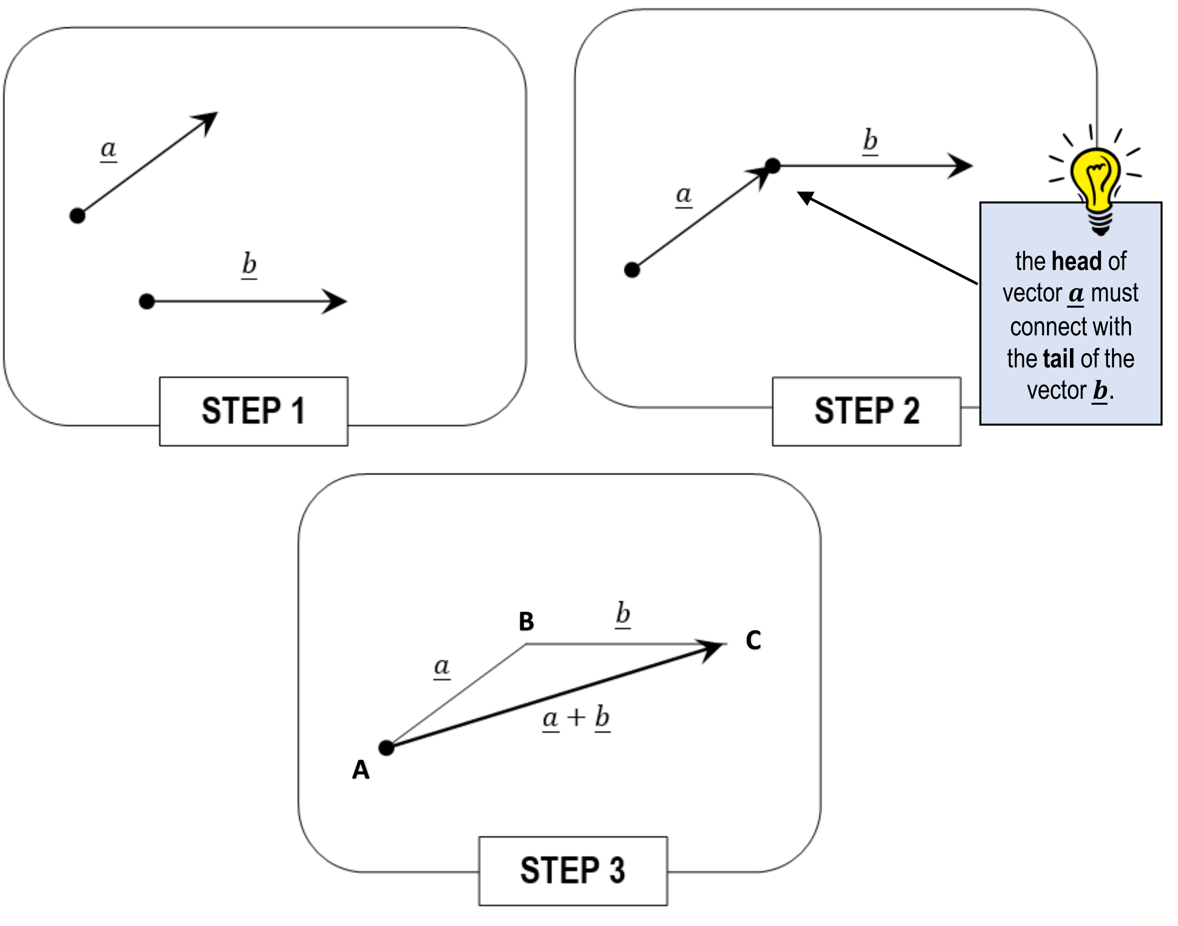
$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{PS}$$

(The resultant $\overrightarrow{PR}$ is the diagonal of a parallelogram)

TRIANGLE LAW OF ADDITION

Also known as Construction Method.

The Triangle Law states that if $\underline{\boldsymbol{a}}$ is represented by $\overrightarrow{\boldsymbol{AB}}$ and $\underline{\boldsymbol{b}}$ by $\overrightarrow{\boldsymbol{BC}}$, then the resultant of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ is represented by $\overrightarrow{\boldsymbol{AC}}$ in a triangle ABC.



$$\overrightarrow{\boldsymbol{AC}} = \overrightarrow{\boldsymbol{AB}} + \overrightarrow{\boldsymbol{BC}}$$

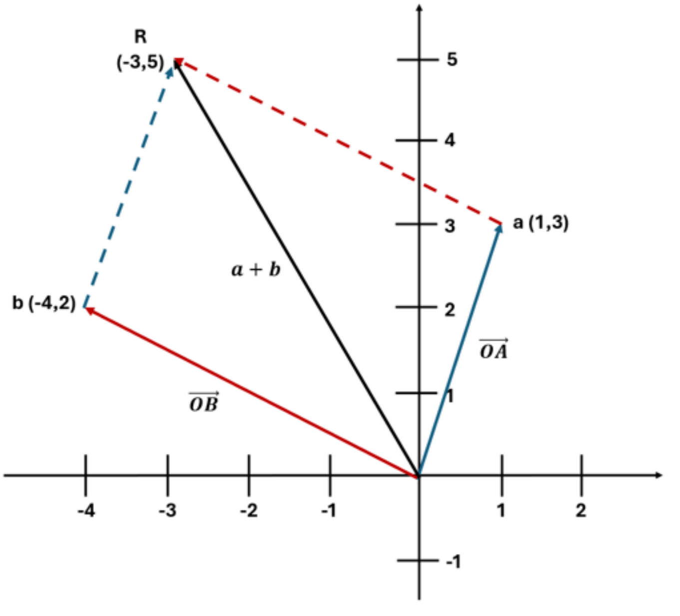
(The resultant vector $\overrightarrow{AC}$ is vector drawn from tail A to head C)

EXAMPLE 4

If $\boldsymbol{a} = \overrightarrow{\boldsymbol{OA}} = (\mathbf{1}, \mathbf{3})$ and $\boldsymbol{b} = \overrightarrow{\boldsymbol{OB}} = (-\mathbf{4}, \mathbf{2})$. Find $\boldsymbol{a} + \boldsymbol{b}$ and draw by using Parallelogram Method and Triangle Method.

Solutions: (using Parallelogram Method)

$$\begin{aligned} \boldsymbol{a} + \boldsymbol{b} &= \overrightarrow{\boldsymbol{OA}} + \overrightarrow{\boldsymbol{OB}} \\ &= (\mathbf{1}, \mathbf{3}) + (-\mathbf{4}, \mathbf{2}) \\ &= (-\mathbf{3}, \mathbf{5}) \end{aligned}$$

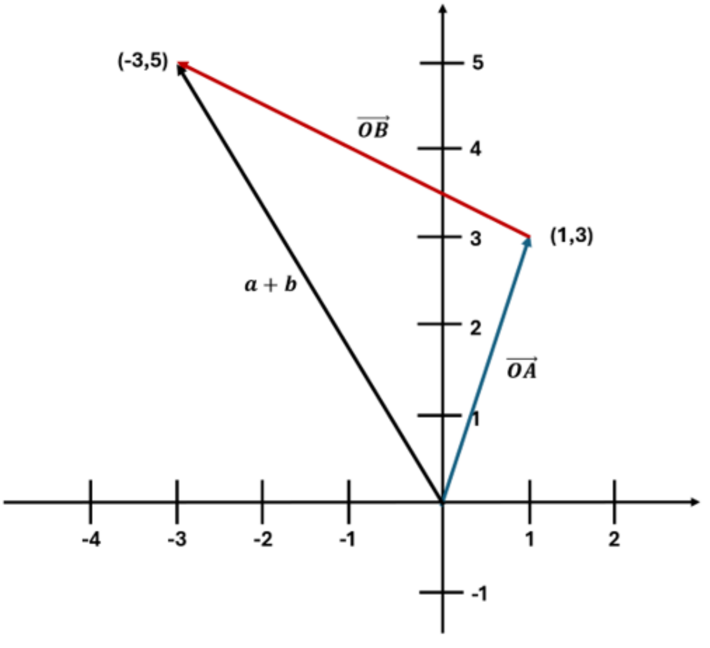


Step:

1. Draw a coordinate system on the graph.
2. Locate the initial point of $\boldsymbol{a}$ at the origin $(\mathbf{0}, \mathbf{0})$. Draw a line segment from the origin to the point $(\mathbf{1}, \mathbf{3})$ to represent $\overrightarrow{\boldsymbol{OA}}$.
3. Locate the initial point of $\boldsymbol{b}$ at the origin $(\mathbf{0}, \mathbf{0})$. Draw a line segment from the origin to the point $(-\mathbf{4}, \mathbf{2})$ to represent $\overrightarrow{\boldsymbol{OB}}$.
4. Complete the parallelogram by drawing the parallel lines.
5. Draw the **resultant vector** by drawing the diagonal of the parallelogram connecting the origin $(\mathbf{0}, \mathbf{0})$ to the terminal points of the additional vectors drawn, $\boldsymbol{a} + \boldsymbol{b}$.

Solutions: (using Triangle Method)

$$\begin{aligned} \boldsymbol{a} + \boldsymbol{b} &= \overrightarrow{\boldsymbol{OA}} + \overrightarrow{\boldsymbol{OB}} \\ &= (\mathbf{1}, \mathbf{3}) + (-\mathbf{4}, \mathbf{2}) \\ &= (-\mathbf{3}, \mathbf{5}) \end{aligned}$$



Step:

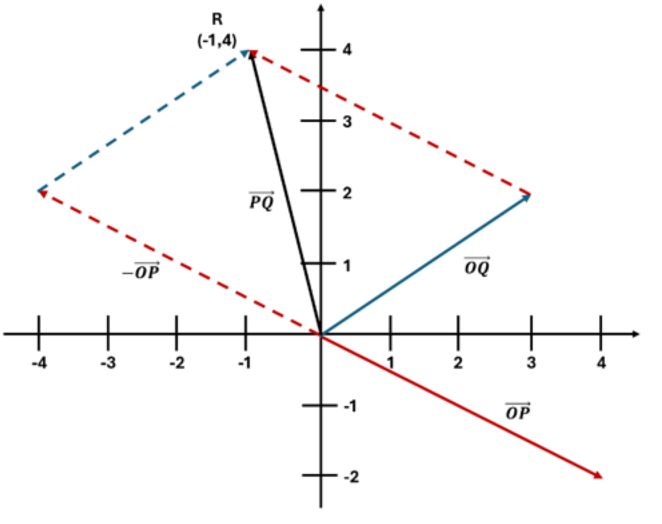
1. Draw a coordinate system on the graph.
2. Locate the initial point of $\boldsymbol{a}$ at the origin $(\mathbf{0}, \mathbf{0})$. Draw a line segment from the origin to the point $(\mathbf{1}, \mathbf{3})$ to represent $\overrightarrow{\boldsymbol{OA}}$.
3. Starting from the head of $\overrightarrow{\boldsymbol{OA}}$, draw $\overrightarrow{\boldsymbol{OB}}$ by move 4 units to the left and 2 units up. Draw a line segment to represent $\overrightarrow{\boldsymbol{OB}}$.
4. Complete the triangle by joining the tail of $\overrightarrow{\boldsymbol{OA}}$ to the head of $\overrightarrow{\boldsymbol{OB}}$.
5. The **resultant vector** $\boldsymbol{a} + \boldsymbol{b}$ is drawn directly from the tail of $\overrightarrow{\boldsymbol{OA}}$ to the head of $\overrightarrow{\boldsymbol{OB}}$.

EXAMPLE 5

Given vector $\overrightarrow{OP} = 4i - 2j$ and $\overrightarrow{OQ} = 3i + 2j$. Draw $\overrightarrow{PQ}$ by using Parallelogram Method and Triangle Method

Solutions: (using Parallelogram Method)

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= -\overrightarrow{OP} + \overrightarrow{OQ} \\ &= -(4i - 2j) + (3i + 2j) \\ &= -4i + 2j + 3i + 2j \\ &= -i + 4j\end{aligned}$$

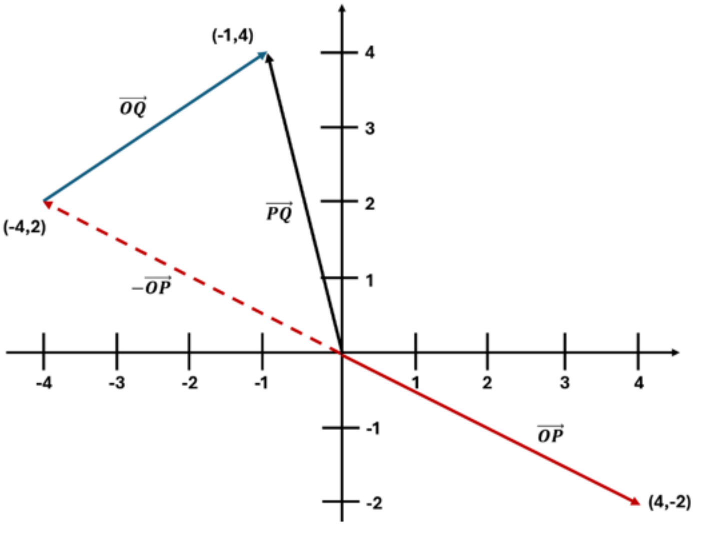


Step:

1. Draw a coordinate system on the graph.
2. Start by drawing the vector $\overrightarrow{OP}$ from the origin $O(0,0)$ to point $P(4,-2)$.
3. Next, draw the vector $\overrightarrow{OQ}$ from the origin $O(0,0)$ to point $Q(3,2)$.
4. Draw the vector $-\overrightarrow{OP}$ (negative vector) by changing its direction to the opposite direction, which is flipping, from the origin $(0,0)$ to point $(-4,2)$.
5. Construct the parallelogram by drawing the parallel lines to reach up R.
6. Draw the **resultant vector** by drawing the diagonal of the parallelogram, which represents $\overrightarrow{PQ}$.

Solutions: (using Triangle Method)

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$
$$= -\overrightarrow{OP} + \overrightarrow{OQ}$$
$$= -(4i - 2j) + (3i + 2j)$$
$$= -4i + 2j + 3i + 2j$$
$$= -i + 4j$$



Step:

1. Draw a coordinate system on the graph.
2. Locate the initial point of vector $\overrightarrow{OP}$ at the origin $(0, 0)$. Draw a line segment from the origin to point $P(4, -2)$.
3. Draw the vector $-\overrightarrow{OP}$ (negative vector) by changing its direction to the opposite direction, which is flipping, from the origin $(0, 0)$ to point $(-4, 2)$.
4. Starting from the head of $-\overrightarrow{OP}$, draw $\overrightarrow{OQ}$ by move 3 units to the right and 2 units up. Draw a line segment to represent $\overrightarrow{OQ}$.
5. Complete the triangle by joining the tail of $-\overrightarrow{OP}$ to the head of $\overrightarrow{OQ}$.
6. The **resultant vector** $\overrightarrow{PQ}$ is drawn directly from the tail of $-\overrightarrow{OP}$ to the head of $\overrightarrow{OQ}$.

EXERCISE

a) Given vector $\vec{P} = 3i + 8j$ and $\vec{Q} = -2i - 3j$. Find $\vec{P} + \vec{Q}$ and draw by using Parallelogram Method and Triangle Method.

03 Scalar (Dot) Product of Two Vectors

1. THE PROPERTIES OF SCALAR PRODUCT
2. THE ANGLE OF SCALAR PRODUCT BETWEEN TWO VECTORS

THE PROPERTIES OF SCALAR PRODUCT

SCALAR PRODUCT

Scalar Product also known as the **Dot Product** of vectors. Scalar product is a multiplication operation on vectors expressed in terms of unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ along the $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ directions.

$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}}$ is the scalar product between two vectors $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ results in **<u>scalar quantity</u>**.

Lets, consider two vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, the scalar product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is represented as $\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}}$ and defined as :

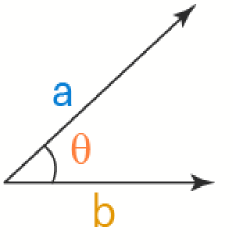
$$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = \boldsymbol{a_1 b_1 + a_2 b_2 + a_3 b_3}$$

Another important method for finding $\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}}$, involving the angle $\boldsymbol{\theta}$ between the two vectors.

- The scalar product is equal to the product of the magnitudes of the two vectors $|\underline{\boldsymbol{a}}|$ and $|\underline{\boldsymbol{b}}|$ and the cosine of the $\boldsymbol{\theta}$ between them.
- Then, the scalar product of two vectors is given by:

$$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = |\underline{\boldsymbol{a}}||\underline{\boldsymbol{b}}| \cos \boldsymbol{\theta},$$

where θ is the angle (degrees) between $\underline{a}$ and $\underline{b}$



a . b = |a||b| cos θ

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EXAMPLE 6

Let the vectors $\boldsymbol{a} = \mathbf{4}\boldsymbol{i} + \mathbf{3}\boldsymbol{j} + \mathbf{7}\boldsymbol{k}$ and $\boldsymbol{b} = \mathbf{2}\boldsymbol{i} + \mathbf{5}\boldsymbol{j} + \mathbf{4}\boldsymbol{k}$.
Find the value of the scalar product.

Solutions:

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= (\mathbf{4}\boldsymbol{i} + \mathbf{3}\boldsymbol{j} + \mathbf{7}\boldsymbol{k}) \cdot (\mathbf{2}\boldsymbol{i} + \mathbf{5}\boldsymbol{j} + \mathbf{4}\boldsymbol{k}) \\ &= 4(2) + 3(5) + 7(4) \\ &= 8 + 15 + 28 \\ &= 51 \end{aligned}$$

EXAMPLE 7

Let the vectors $\boldsymbol{a} = \mathbf{3}\boldsymbol{i} - \mathbf{2}\boldsymbol{j} + \mathbf{5}\boldsymbol{k}$ and $\boldsymbol{b} = \mathbf{4}\boldsymbol{i} + \boldsymbol{j} + \mathbf{2}\boldsymbol{k}$.
Find the value of the scalar product.

Solutions:

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= (\mathbf{3}\boldsymbol{i} - \mathbf{2}\boldsymbol{j} + \mathbf{5}\boldsymbol{k}) \cdot (\mathbf{4}\boldsymbol{i} + \boldsymbol{j} + \mathbf{2}\boldsymbol{k}) \\ &= 3(4) + (-2)(1) + 5(2) \\ &= 12 - 2 + 10 \\ &= 20 \end{aligned}$$

EXAMPLE 8

Find $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$, if $\boldsymbol{a} = \mathbf{5}\boldsymbol{i} - \mathbf{4}\boldsymbol{j}$, $\boldsymbol{b} = \mathbf{7}\boldsymbol{i} + \mathbf{8}\boldsymbol{j}$ and
$\boldsymbol{c} = \mathbf{3}\boldsymbol{i} - \mathbf{2}\boldsymbol{j}$.

Solutions:

$$\begin{aligned} \boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) &= (\mathbf{5}\boldsymbol{i} - \mathbf{4}\boldsymbol{j}) \cdot ((\mathbf{7}\boldsymbol{i} + \mathbf{8}\boldsymbol{j}) + (\mathbf{3}\boldsymbol{i} - \mathbf{2}\boldsymbol{j})) \\ &= (5i - 4j) \cdot (10i + 6j) \\ &= 5(10) + (-4)(6) \\ &= 50 - 24 \\ &= 26 \end{aligned}$$

THE ANGLE OF SCALAR PRODUCT BETWEEN TWO VECTORS

CALCULATE THE ANGLE BETWEEN TWO VECTORS

To find the angle between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ using Scalar Product method, we use:

$$\boldsymbol{\theta} = \mathbf{cos}^{-1}\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

EXAMPLE 9

Calculate the angle between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ if $|\boldsymbol{a}| = \mathbf{1}$, $|\boldsymbol{b}| = \mathbf{2}$, and their dot product is $\boldsymbol{a}\cdot\boldsymbol{b} = \mathbf{1}$.

Solutions:

Let us assume that the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{\theta}$.

Then we have: $$\boldsymbol{\theta} = \mathbf{cos}^{-1}\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

$$\begin{aligned} \therefore \boldsymbol{\theta} &= \cos^{-1}\frac{1}{(1)(2)} \\ &= \cos^{-1} 0.5 \\ &= 60^\circ \end{aligned}$$

EXAMPLE 10

Find the angle between the two vectors $\boldsymbol{a} = \boldsymbol{2i} + \boldsymbol{3j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{5i} - \boldsymbol{2j} + \boldsymbol{3k}$ using Scalar Product method.

Solutions:

Step 1: Calculate the Scalar Product of a and b based on their corresponding components:

$$
\begin{aligned}
\boldsymbol{a} \cdot \boldsymbol{b} &= (2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) \cdot (5\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}) \\
&= 2(5) + 3(-2) + 1(3) \\
&= 10 + (-6) + 3 \\
&= 7
\end{aligned}
$$

Step 2: Calculate the magnitudes of a and b :

$$
\begin{aligned}
|\boldsymbol{a}| &= \sqrt{2^2 + 3^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14} \\
|\boldsymbol{b}| &= \sqrt{5^2 + (-2)^2 + 3^2} = \sqrt{25 + 4 + 9} = \sqrt{38}
\end{aligned}
$$

Step 3: Substitute these values into the Scalar Product formula to find the θ :

$$
\boldsymbol{\theta} = \cos^{-1} \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}
$$

$$
\begin{aligned}
\therefore \boldsymbol{\theta} &= \cos^{-1} \frac{7}{\sqrt{14}\sqrt{38}} \\
&= \cos^{-1} 0.303 \\
&= 72.36°
\end{aligned}
$$

EXERCISE

a) Evaluate the followings:

i. $(5\mathbf{i} - \mathbf{j}) \cdot (2\mathbf{i} + 3\mathbf{j})$

ii. $(-4\mathbf{i} + 7\mathbf{j}) \cdot (8\mathbf{j})$

iii. $(\mathbf{i} - 3\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j})$

iv. $(2\mathbf{i} - \mathbf{j}) \cdot (4\mathbf{i} - \mathbf{j})$

b) Given two vectors $\vec{\boldsymbol{A}} = \boldsymbol{2i} - \boldsymbol{3j}$ and $\vec{\boldsymbol{B}} = \boldsymbol{i} + \boldsymbol{2j}$ **.** Find the angle between $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ using Scalar Product.

04 Vector (Cross) Product of Two Vectors

1. DEFINITION OF VECTOR PRODUCT

2. THE PROPERTIES OF VECTOR PRODUCT

2. CALCULATION OF THE AREA TRIANGLE AND PARALLELOGRAM

DEFINITION OF VECTOR PRODUCT

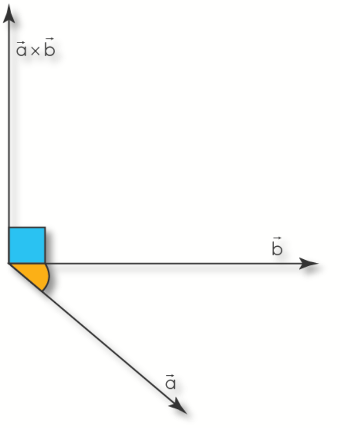
VECTOR PRODUCT

Vector Product also known as the **Cross Product** of vectors

Two vectors in three-dimensional space can be multiplied using the "**Cross Product**" results in a **vector quantity**. The cross product of the two vectors is given by the formula:

$$|\boldsymbol{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \boldsymbol{\theta}\, \hat{\boldsymbol{n}}$$

- ***a*** is the magnitude (length) of vector ***a***
- ***b*** is the magnitude (length) of vector ***b***
- ***θ*** is the angle between ***a*** and ***b***
- ***n*** is the unit vector at right angles to both ***a*** and ***b***



$$\vec{a} \times \vec{b} = \vec{c}$$

Where $\vec{a}$ and $\vec{b}$ are two vectors, and $\vec{c}$ is the resultant vector.

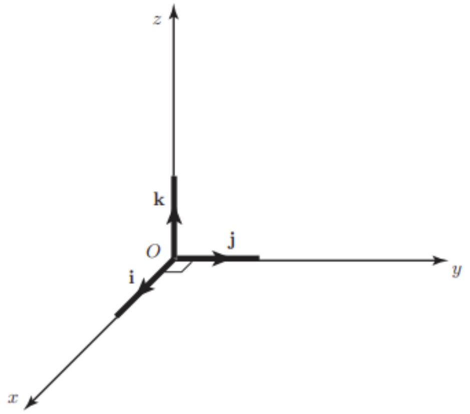
Let's assume that $\vec{a}$ and $\vec{b}$ are two vectors, such that $\vec{a} = a_1 i + a_2 j + a_3 k$ and $\vec{b} = b_1 i + b_2 j + b_3 k$ then by using determinants, we could find the cross product and write the result as the cross-product formula using matrix notation.

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \mathbf{i} & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

THE PROPERTIES OF VECTOR PRODUCT

THE PROPERTIES OF VECTOR PRODUCT



Let, the angle between $\boldsymbol{i}$ and $\boldsymbol{j}$ is $90°$, and sin $90°$ = 1. A vector perpendicular to both $\boldsymbol{i}$ and $\boldsymbol{j}$ is $\boldsymbol{k}$.

Therefore;

$$\begin{aligned} \boldsymbol{i} \,.\, \boldsymbol{j} &= |\mathbf{i}||\mathbf{j}| \sin 90° \, \boldsymbol{k} \\ &= (\mathbf{1})(\mathbf{1})(\mathbf{1})\boldsymbol{k} \\ &= \boldsymbol{k} \end{aligned}$$

EXAMPLE 11

Let the vectors $\boldsymbol{a} = (\mathbf{3}, \mathbf{5}, -\mathbf{7})$ and $\boldsymbol{b} = (\mathbf{2}, -\mathbf{6}, \mathbf{4})$.
Find the value $\boldsymbol{a} \times \boldsymbol{b}$ using Cross Product method.

Solutions:

Let the vectors $\boldsymbol{a} = \mathbf{3}\boldsymbol{i} + \mathbf{5j} - \mathbf{7k}$ and $\boldsymbol{b} = \mathbf{2i} - \mathbf{6}\boldsymbol{j} + \mathbf{4k}$.

By using determinants, we could find the Cross Product and write the result as the Cross Product formula using the following matrix notation.

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \mathbf{i} & \boldsymbol{j} & \boldsymbol{k} \\ \boldsymbol{a_1} & \boldsymbol{b_1} & \boldsymbol{c_1} \\ \boldsymbol{a_2} & \boldsymbol{b_2} & \boldsymbol{c_2} \end{vmatrix}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{b_1} & \boldsymbol{c_1} \\ \boldsymbol{b_2} & \boldsymbol{c_2} \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} \boldsymbol{a_1} & \boldsymbol{c_1} \\ \boldsymbol{a_2} & \boldsymbol{c_2} \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} \boldsymbol{a_1} & \boldsymbol{b_1} \\ \boldsymbol{a_2} & \boldsymbol{b_2} \end{vmatrix} \boldsymbol{k}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 5 & -7 \\ 2 & -6 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & -7 \\ -6 & 4 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 3 & -7 \\ 2 & 4 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 3 & 5 \\ 2 & -6 \end{vmatrix} \boldsymbol{k}$$

$$= |20 - 42|\boldsymbol{i} - |12 - (-14)|\boldsymbol{j} + |-18 - 10|\boldsymbol{k}$$

$$= -22\boldsymbol{i} - 26\boldsymbol{j} - 28\boldsymbol{k}$$

EXAMPLE 12

Let the vectors $\boldsymbol{a} = (\mathbf{2}, -\mathbf{4}, \mathbf{4})$ and $\boldsymbol{b} = (\mathbf{4}, \mathbf{0}, \mathbf{3})$.
Find the angle between them using Cross Product

Solutions:

Step 1: Calculate the Cross Product of a and b using matrix notation:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -4 & 4 \\ 4 & 0 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} -4 & 4 \\ 0 & 3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 2 & 4 \\ 4 & 3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 2 & -4 \\ 4 & 0 \end{vmatrix} \boldsymbol{k}$$

$$= |-12 - 0|\boldsymbol{i} - |6 - 16|\boldsymbol{j} + |0 - (-16)|\boldsymbol{k}$$

$$= -12\boldsymbol{i} + 10\boldsymbol{j} + 16\boldsymbol{k}$$

Step 2: Find the magnitude of the Cross Product:

$$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{(-12)^2 + (10)^2 + 16^2}$$

$$= \sqrt{144 + 100 + 256}$$

$$= \sqrt{500} = 22.36$$

Step 3: Find the $\boldsymbol{\theta}$:

$$|\boldsymbol{a}| = \sqrt{2^2 + (-4)^2 + 4^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$$

$$|\boldsymbol{b}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{16 + 0 + 9} = \sqrt{25} = 5$$

$$\boldsymbol{\theta = \sin^{-1} \frac{|a \times b|}{|a||b|}}$$

$$\therefore \theta = \sin^{-1} \frac{22.36}{(6)(5)}$$

$$= \sin^{-1} 0.745 = 48.16°$$

EXERCISE

a) Find the cross product $\boldsymbol{u} \times \boldsymbol{v}$ of the vectors $\boldsymbol{u} = (\mathbf{2}, \mathbf{0}, \mathbf{0})$ and $\boldsymbol{v} = (\mathbf{2}, \mathbf{2}, \mathbf{0})$.

b) Find the cross product $\boldsymbol{u} \times \boldsymbol{v}$ of the vectors $\boldsymbol{u} = \mathbf{2}\boldsymbol{i} + \mathbf{3}\boldsymbol{j}$ and $\boldsymbol{v} = \boldsymbol{j} + \mathbf{2}\boldsymbol{k}$.

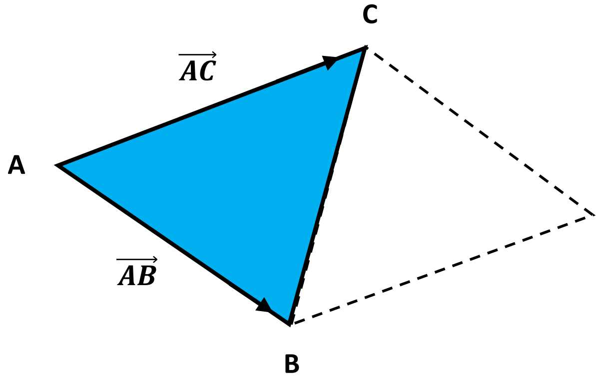
c) Use the cross product to find the value of $\boldsymbol{sin\,\theta}$, where $\boldsymbol{\theta}$ is the angle between $\boldsymbol{u} \times \boldsymbol{v}$ of the vectors $\boldsymbol{u} = \boldsymbol{i} + \mathbf{2}\boldsymbol{j} + \mathbf{3}\boldsymbol{k}$ and $\boldsymbol{v} = \mathbf{3}\boldsymbol{i} - \mathbf{2}\boldsymbol{j} + \boldsymbol{k}$.

d) Given two vectors $\vec{\boldsymbol{A}} = (\mathbf{1}, \mathbf{1}, \mathbf{1})$ and $\vec{\boldsymbol{B}} = (\mathbf{1}, \mathbf{2}, \mathbf{3})$ **.** Find the angle $\boldsymbol{\theta}$ between $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ using Cross Product.

e) Find the dot product and cross product for vectors $\vec{\boldsymbol{P}}$ and $\vec{\boldsymbol{Q}}$, where $\mathbf{3i} - \mathbf{4j} + \mathbf{2k}$ and $\mathbf{3}\boldsymbol{i} - \mathbf{5}\boldsymbol{j} - \boldsymbol{k}$ respectively.

CALCULATION OF THE AREA TRIANGLE AND PARALLELOGRAM

AREA OF TRIANGLE IN VECTOR FORM



It is known that the area of a triangle is half the area of a parallelogram.

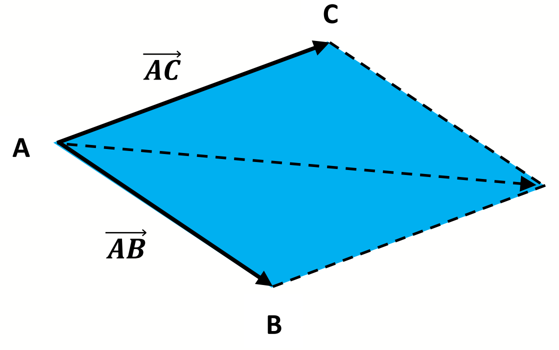
The area of a triangle formed with the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as two of its sides, is given by **half of the magnitude of the cross product of the two vectors**.

The formula for the area of a triangle is:

$$\textit{Area of triangle}, A = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

CALCULATION OF THE AREA TRIANGLE AND PARALLELOGRAM

AREA OF PARALELLOGRAM IN VECTOR FORM



The area of a parallelogram in vector form is geometrically **the magnitude of the cross product of two vectors**.

The area of a parallelogram whose adjacent sides are the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is $|\overrightarrow{AB} \times \overrightarrow{AC}|$.

$$\textit{Area of paralellogram}, A = |\overrightarrow{AB} \times \overrightarrow{AC}|$$

EXAMPLE 13

Given the vectors $\boldsymbol{a} = (\mathbf{1}, \mathbf{2}, \mathbf{3})$ and $\boldsymbol{b} = (\mathbf{4}, \mathbf{5}, \mathbf{6})$.
Find the area of the triangle determined by these two vectors.

Solutions:

Step 1: Find the Cross Product of $\boldsymbol{a} \times \boldsymbol{b}$.

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \mathrm{i} & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \boldsymbol{k}$$

$$= |12 - 15|\boldsymbol{i} - |6 - 12|\boldsymbol{j} + |5 - 8|\boldsymbol{k}$$

$$= -3\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$$

$$= (-3, 6, -3)$$

Step 2: Calculate the area of triangle.

$$A = \frac{1}{2}|a \times b|$$

$$= \frac{1}{2}\sqrt{(-3)^2 + 6^2 + (-3)^2}$$

$$= \frac{1}{2}\sqrt{9 + 36 + 9}$$

$$= \frac{1}{2}\sqrt{54}$$

$$= \frac{1}{2}(7.348)$$

$$= 3.674$$

EXAMPLE 14

Given the vectors $\boldsymbol{u} = (\mathbf{1}, -\mathbf{2}, \mathbf{5})$ and $\boldsymbol{v} = (\mathbf{2}, \mathbf{0}, -\mathbf{1})$.
Find the area of the paralellogram enclosed by these two vectors.

Solutions:

Step 1: Find the Cross Product of $\boldsymbol{u} \times \boldsymbol{v}$.

$$
\begin{aligned}
\boldsymbol{u} \times \boldsymbol{v} &= \begin{vmatrix} \mathrm{i} & j & k \\ 1 & -2 & 5 \\ 2 & 0 & -1 \end{vmatrix} \\
&= \begin{vmatrix} -2 & 5 \\ 0 & -1 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 1 & 5 \\ 2 & -1 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 1 & -2 \\ 2 & 0 \end{vmatrix} \boldsymbol{k} \\
&= |2 - 0| \boldsymbol{i} - |-1 - 10| \boldsymbol{j} + |0 - (-4)| \boldsymbol{k} \\
&= 2\boldsymbol{i} + 11\boldsymbol{j} + 4\boldsymbol{k} \\
&= (2, 11, 4)
\end{aligned}
$$

Step 2: Calculate the area of parallelogram.

$$
\begin{aligned}
A &= |u \times v| \\
&= \sqrt{2^2 + 11^2 + 4^2} \\
&= \sqrt{4 + 121 + 16} \\
&= \sqrt{141} \\
&= 11.87
\end{aligned}
$$

VECTOR & SCALAR

FOR MALAYSIAN POLYTECHNIC STUDENTS

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